CONVECTIVE MOTIONS IN A POROUS MEDIUM HEATED FROM BELOW

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Convective motions in a porous medium filling a horizontal cylinder with a cross section of arbitrary shape and heated from below are studied. The small-parameter method is used to obtain infinitely many stationary motions forming a one-parameter family. For small values of the parameter, all of these motions are stable with respect to small perturbations. The article also discusses the case of heating which is not strictly vertical. It is found that in this case only one stationary motion is stable.

1. We consider a closed volume filled with a porous medium saturated with a liquid. The equations of heat convection in a porous medium have the form [1]

$$\frac{1}{\varepsilon}\frac{\partial \mathbf{v}}{\partial t} = -\frac{1}{\rho_f}\nabla p + g\beta_f T\gamma - \frac{\nu}{K}\mathbf{v}; \qquad (1.1)$$

$$b\frac{\partial T}{\partial t} + \mathbf{v}\nabla T = \chi\Delta T;$$

$$\operatorname{div}\mathbf{v} = 0.$$

Here **v** is the filtration velocity; T is the temperature, measured from some average value; p is the pressure; ρ is the density; g is the acceleration of gravity; β is the coefficient of thermal expansion; ν is the kinematic viscosity; K is the permeability; ε is the porosity; $b=(\rho C_p)_m/(\rho C_p)_f$; $\chi = \varkappa_m/(\rho C_p)_f$; γ is a unit vector directed vertically upward. The subscript f denotes quantities relating to the liquid, and the subscript m denotes quantities relating to the porous medium together with the liquid.

The characteristic time of equalization of the velocity is $\tau_v \sim K/\nu\varepsilon$, and the characteristic time of equalization of the temperature is $\tau_T \sim h^2/\chi_f$, where h is the characteristic dimension of the cavity; χ_f is the thermal diffusivity of the liquid. For the values $\chi_f/\nu \sim 1$, $\varepsilon \sim 0.1$, $h \sim 10$ cm, $K \sim 10^{-4}$ cm^{-2}, which are typical for laboratory conditions, we obtain $\tau_v/\tau_T \cong 10^{-5} \ll 1$, so that we can omit the term involving the derivative with respect to time in the equation of motion, while retaining the analogous term in the equation of heat conduction.

On the surface S bounding the volume the normal component of the filtration velocity, v_n, vanishes, and we are given a temperature distribution such that if there is no motion in the region, a constant vertical temperature gradient A is established (for heating from below we have $A > 0$). Under these conditions, Eqs. (1.1) admit of an equilibrium solution with $\mathbf{v}=0$. For sufficiently large values of A the equilibrium becomes unstable. Let us write the equation for finite disturbances of the equilibrium. We introduce dimensionless variables, selecting the following units: for distance the characteristic dimension h of the cavity, for velocity χ/h, for time bh^2/χ , for pressure $\nu\chi/K\rho_f$, and for temperature $\sqrt{\nu\chi A/g\beta_f K}$. As a result, we obtain the following system of equations:

$$-\nabla p + cT\gamma - \mathbf{v} = 0;$$

$$\frac{\partial T}{\partial t} + \mathbf{v}\nabla T = \Delta T + c\mathbf{v}\gamma;$$

$$\operatorname{div}\mathbf{v} = 0$$

with the boundary conditions

$$T|_S = v_n|_S = 0.$$

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Here $c=\sqrt{R}$; $R=\frac{g\beta Ah^2K}{\nu\chi}$ is the analog of the Rayleigh number.

We consider a cavity in the form of a horizontal cylinder with cross section D bounded by the contour Γ. We introduce the system of coordinates x, y, z, in which the z axis is directed parallel to a generator of the cylinder and the y axis is directed vertically upward.

We shall confine ourselves to consideration of plane motion, for which v_z and all the quantities are independent of z.

We introduce the stream function by means of the relations

$$v_x=-\frac{\partial\psi}{\partial y};\quad v_y=\frac{\partial\psi}{\partial x}.$$

In terms of the stream function, the equations and the boundary conditions take the form

$$\Delta\psi-c\frac{\partial T}{\partial x}=0;\quad \frac{\partial T}{\partial t}+D(\psi,T)=\Delta T+c\frac{\partial\psi}{\partial x}; \tag{1.2}$$

$$T|_\Gamma=\psi|_\Gamma=0. \tag{1.3}$$

Here $D(\psi, T)$ is the Jacobian with respect to the variables x, y.

The equations for small perturbations of the equilibrium are obtained from (1.2) by omitting the nonlinear term. For these equations the principle of monotonicity of perturbations can be shown in the usual way [2]. Therefore for critical perturbations we shall have the equations

$$\Delta\psi-c_*\frac{\partial T}{\partial x}=0;\quad \Delta T+c_*\frac{\partial\psi}{\partial x}=0 \tag{1.4}$$

with the boundary conditions (1.3) (c_* is the critical value of the parameter c). It can be seen that if ψ_1, T_1 is a solution of the problem (1.4)-(1.3), then

$$\psi_2=T_1;\quad T_2=-\psi_1 \tag{1.5}$$

is also a solution and the two solutions are linearly independent. Thus, all the critical numbers c_* are at least doubly degenerate. A higher multiplicity of degeneracy evidently can exist if there is a special symmetry in the region, and therefore we shall hereafter assume that the lower level is doubly degenerate. In any case, this is true for a region of square shape, for which we can obtain an exact solution of the problem (1.4)-(1.3). Placing the origin at the center of the square and taking half the side of the square as our unit of length, we can write this solution in the form

$$\psi=(\cos\gamma_1 x+\cos\gamma_2 x)\cos\frac{\pi y}{2};$$

$$T=(\sin\gamma_1 x+\sin\gamma_2 x)\cos\frac{\pi y}{2};\quad \gamma_{1,2}=-\pi\frac{\sqrt{2}\pm 1}{2}.$$

The second solution is obtained from this by the transformation (1.5). The corresponding critical number is $c_*=\pi\sqrt{2}$.

2. Now let us study the stationary solutions of the system (1.2)-(1.3), which branch off from the trivial solution at the point $c=c_*$. We shall try to find the solution in the form of series in fractional powers of $(c-c_*)$. These series can be obtained by standard methods of branching theory [3]; however, because of the double degeneracy of the linear problem, we cannot show the convergence of the resulting series, unlike the case of convection in a homogeneous liquid [4], in which the first critical number is not degenerate. The series themselves can more conveniently be obtained by the method of undetermined coefficients.

First we rewrite the problem in a somewhat different form. We introduce the function $\varphi=\psi+iT$. From (1.2)-(1.3) we obtain a boundary-value problem for the function φ:

$$\frac{1}{2}\frac{\partial(\varphi-\bar\varphi)}{\partial t}-\frac{1}{2}D(\varphi,\bar\varphi)=\Delta\varphi+ic\frac{\partial\varphi}{\partial x},\ \varphi|_\Gamma=0. \tag{2.1}$$

Hereafter we shall not write out the boundary conditions. A bar above a symbol will denote a complex conjugate.

As in the theory of convection in a homogeneous liquid [2], we can show that when $c<c_*$, there are no stationary solutions. It is therefore natural to try to find a solution for $c>c_*$ in the form of series in fractional powers of $(c-c_*)$, where the exponents are multiples of $1/n$ with even n. As is known [2, 4], in the case of convection in a homogeneous liquid the solution can be represented in the form of series with $n=2$. Since in our case the nonlinearity is also quadratic, we shall try to find the stationary solution in the form of a series

$$\varphi(x, y) = \sum_{n=1}^{\infty} \varphi_n(x, y)\lambda^n; \quad \lambda = \sqrt{2(c-c_*)}. \tag{2.2}$$

The possibility of representing the solution in the form of a series in other powers of λ will not be considered further.

Substituting (2.2) into (2.1) and collecting terms with equal powers of λ, we obtain an infinite system of linked equations,

$$L\varphi_n = i\frac{\partial\varphi_{n-2}}{\partial x} + \sum_{j=1}^{n-1} D(\varphi_j, \bar{\varphi}_{n-j}),\ n=1,\ 2,\ \ldots$$

Here $L=-2[\Delta + ic_*(\partial/\partial x)]$, and quantities with nonpositive indices are assumed to be identically zero.

For the first order we have a homogeneous problem $L\varphi_1 = 0$. We write the solution in the form $\varphi_1 = \varepsilon\exp(i\eta)f$, where $\eta \in (-\pi/2, \pi/2]$, and f is the solution of the problem of critical perturbations normed by the condition

$$i\left\langle \bar{f}\frac{\partial f}{\partial x}\right\rangle = c_*^{-1}\langle|\nabla f|^2\rangle = 1.$$

The pointed brackets denote integration with respect to D. The amplitude ε and the phase η must be determined from the conditions of solvability of the higher-order equations. The operator adjoint to L coincides with its complex conjugate, and therefore the condition for solvability in the n-th-order case has the form

$$i\left\langle \bar{f}\frac{\partial\varphi_{n-2}}{\partial x}\right\rangle + \sum_{j=1}^{n-1}\langle\bar{f}D(\varphi_j, \bar{\varphi}_{n-j})\rangle = 0. \tag{2.3}$$

In the equation for φ_2

$$L\varphi_2 = \varepsilon^2 D(f, \bar{f}) \tag{2.4}$$

the phase η does not appear. Therefore φ_2 depends only on the amplitude ε. The condition for solvability of Eq. (2.4), $\langle\bar{f}D(f\cdot\bar{f})\rangle = 0$, is identically satisfied, i.e., ε and η are undefined for this order. For $n=3$ we obtain from (2.3), after some simple transformations, the equation

$$\langle\bar{\varphi}_2 D(f, \bar{f})\rangle = 1.$$

From this relation we can determine ε except for its sign, which corresponds to the two possible directions of circulation for a given η.

It can be shown that for any η the solvability condition (2.3) can be satisfied for all orders.

Thus, when $c > c_*$, we can construct infinitely many stationary solutions of the system (1.2) in the form of series in λ. These solutions differ in the value of the parameter η. The questions concerning the convergence of the expansions and the existence of other solutions remain open.

3. It is of interest to investigate the stability of the solutions constructed in Sec. 2 for small perturbations. Denoting a perturbation by a prime, we obtain from (2.1), after linearization, the equation

$$\frac{\partial(\bar{\varphi}' - \varphi')}{\partial t} + D(\varphi, \bar{\varphi}') + D(\varphi', \bar{\varphi}) = L\varphi' - i\lambda^2\frac{\partial\varphi'}{\partial x}. \tag{3.1}$$

We consider the "normal" perturbations:

$$\varphi'(x, y, t) = \chi(x, y)\exp(\sigma t) + \varkappa(x, y)\exp(\bar{\sigma}t). \tag{3.2}$$

Substituting (3.2) into (3.1), we obtain a system of equations for the amplitudes χ and ϰ:

$$L\chi = i\lambda^2\frac{\partial\chi}{\partial x} + D(\varphi, \bar{\varkappa}) + D(\chi, \bar{\varphi}) + \sigma(\bar{\varkappa} - \chi); \tag{3.3}$$

$$L\varkappa = i\lambda^2\frac{\partial\varkappa}{\partial x} + D(\varphi, \bar{\chi}) + D(\varkappa, \bar{\varphi}) + \sigma(\bar{\chi} - \varkappa).$$

At the critical point (for $\lambda = 0$) all the decrements are real and negative except for one which is zero. For a determination of stability near the critical point, it is important to know the behavior for small values of λ of precisely that perturbation whose decrement vanishes when $\lambda = 0$. The amplitude and decrement of this perturbation can be represented in the form of series in λ:

$$\chi = \sum_{k=0}^{\infty} \chi_k \lambda^k; \quad \varkappa = \sum_{k=0}^{\infty} \varkappa_k \lambda^k; \quad \sigma = \sum_{k=1}^{\infty} \sigma_k \lambda^k. \tag{3.4}$$

Substituting the series (3.4) into (3.3), we obtain for zero order

$$L\chi_0 = 0; \; L\varkappa_0 = 0.$$

Without loss of generality, we can write $\varkappa_0 = \varphi_1$; $\chi_0 = a\varphi_1$, where a must be determined in the subsequent orders. We write the equations of the first order:

$$L\chi_1 = (1+a)D(\varphi_1, \; \overline{\varphi}_1) + \sigma_1(\overline{\varphi}_1 - a\varphi_1);$$
$$L\varkappa_1 = (1+\overline{a})D(\varphi_1, \; \overline{\varphi}_1) + \overline{\sigma}_1(\overline{a}\overline{\varphi}_1 - \varphi_1).$$

From the conditions of solvability of these equations it follows that $\sigma_1 = 0$; therefore,

$$\chi_1 = (1+a)\varphi_2; \; \varkappa_1 = (1+\overline{a})\varphi_2.$$

The solvability conditions for the equations of second order lead to an algebraic system for determining σ_2 and a:

$$\sigma_2(\beta \exp(-2i\eta) - a\alpha) = 1 + a; \; \sigma_2(a\beta \; \exp(2i\eta) - \alpha) = 1 + a;$$
$$\alpha = \langle |f|^2 \rangle; \; \beta = \langle f^2 \rangle. \tag{3.5}$$

The system (3.5) admits of two solutions:

$$a^{(1)} = \frac{\alpha + \beta \exp(-2i\eta)}{\alpha + \beta \exp(2i\eta)}; \quad \sigma_2^{(1)} = -2\frac{\alpha + \beta \cos 2\eta}{\alpha^2 - \beta^2};$$
$$a^{(2)} = -1, \; \sigma_2^{(2)} = 0.$$

We can assume that β is real, since a real β can always be obtained by a suitable choice of f. Considering the perturbations in equilibrium and taking account of the fact that c_* is the first critical number, we can show that $\alpha > |\beta|$. Therefore $\sigma_2^{(1)} < 0$, and the corresponding perturbation is damped. With respect to the second solution, it can be shown that the corresponding decrement remains zero for all orders. The existence of neutral perturbations is a simple corollary of the fact that we have a continuous set of stationary solutions. By differentiating the stationary equations with respect to η, we find that Eqs. (3.3) are satisfied when $\chi = \varkappa = \partial\varphi/\partial\eta$, $\sigma = 0$. The same is true for convection in an infinite plane horizontal layer of homogeneous liquid, where it is possible to have stationary solutions with different periods, and for convection in a spherical cavity, where the fundamental motion has no symmetry with respect to a vertical axis, so that rotation by an arbitrary angle about this axis leads to a new solution.

Thus, all the solutions constructed in Sec. 2 are stable near the threshold with respect to small perturbations, and there is a neutral perturbation.

4. In a real situation, we can select some single solution from the totality of all stationary solutions. This may be a consequence of the small deviation of the conditions of the problem from the conditions considered above. Let us consider the effect produced on the motion by heating which is not strictly vertical. Suppose that at the boundaries of the region we are given a temperature distribution such that under conditions of pure heat conduction in the region we would have a temperature gradient with a vertical component A_v and a horizontal component A_h. On the basis of this gradient, we can establish two Rayleigh numbers, R_v and R_h. We denote by A_* the critical value of A_v when $A_h = 0$. It is convenient to select the following units of measurement: for the stream function $2c_*\chi$, for temperature $2\sqrt{A_v A_*}h$, and for time $bh^2/2c_*\chi$.

Again separating the heat-conduction part from the temperature and introducing the function φ, we reduce the problem to

$$\frac{\partial(\overline{\varphi} - \varphi)}{\partial t} + D(\varphi, \overline{\varphi}) = L\varphi - i\lambda^2 \frac{\partial\varphi}{\partial x} + \xi^3 - \xi^3 \frac{i}{1+\lambda^2} \frac{\partial(\varphi + \overline{\varphi})}{dy}; \tag{4.1}$$
$$L = -c_*^{-1}\left(\Delta + ic_* \frac{\partial}{\partial x}\right); \quad \lambda = \left(\frac{c_v - c_*}{c_*}\right)^{1/2}; \quad \xi = \left(\frac{R_h}{2R_*}\right)^{1/3}.$$

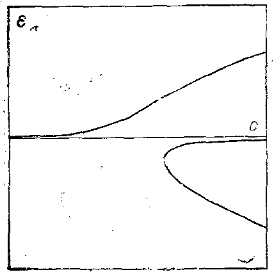

Fig. 1

Considering λ and ξ to be small, we shall try to find stationary solutions of Eq. (4.1) along different rays $\lambda = t\xi$ in the form of series in powers of ξ. For the first power we obtain a homogeneous problem, from which it follows that

$$\varphi_1 = \varepsilon \exp(i\eta) f; \quad \eta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

For the third power we obtain the equation

$$L\varphi_3 = it^2 \frac{\partial \varphi_1}{\partial x} + D(\varphi_1, \bar{\varphi}_2) + D(\varphi_2, \bar{\varphi}_1) - 1. \tag{4.2}$$

Let us denote by ε_v the amplitude we would have for the given vertical heating and for $\xi = 0$. We set

$$\langle \bar{f} \rangle = K \exp(i\vartheta); \ \vartheta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

Then the condition of solvability of Eq. (4.2) can be written in the form

$$\varepsilon^3 - (\varepsilon_v t)^2 \varepsilon + K\varepsilon_v^2 \exp i(\vartheta - \eta) = 0. \tag{4.3}$$

Since in (4.3) the first two terms are real, it follows that $\eta = \vartheta$. Thus the degeneracy is removed, and we find a separated phase value. For determining the amplitude, we obtain a cubic equation which has one real solution for

$$t^2 < t_*^2 = 3\left(\frac{K}{2\varepsilon_v}\right)^{2/3}$$

and three real solutions for $t^2 > t^2_*$. The rays $\lambda = \pm t_* \xi$ are branching lines in the $\lambda - \xi$ plane which separate regions with one solution from regions with three solutions. Fixing ξ and considering the solution along various rays, we can obtain ε as a function of λ; this is qualitatively shown in Fig. 1.

Thus, it is sufficient to have an arbitrarily weak lateral heating in order to obtain a situation in which out of the entire set of solutions we are left with only one or three which behave as in a homogeneous liquid [5].

The stability of the resulting solutions can be investigated in a manner analogous to that of Sec. 3. Without carrying out the computations, which are rather cumbersome in this case, we shall state only the final result:

$$\sigma_0 = \sigma_1 = 0;$$
$$\sigma_2 = \left(\frac{\varepsilon}{\varepsilon_v}\right)^2 \frac{-(\alpha d + \beta\cos 2\vartheta) \pm \sqrt{(\alpha d + \beta\cos 2\vartheta)^2 + (1 - d^2)(\alpha^2 - \beta^2)}}{\alpha^2 - \beta^2}, \tag{4.4}$$

where $d = 2 - t_2(\varepsilon_v/\varepsilon)^2$. Since $\alpha < |\beta|$, the expression under the radical sign in (4.4) is positive, i.e., σ_2 is real. It can be seen that both values are negative only on the upper branch of the three illustrated in Fig. 1. This means that the only stable branch is the one corresponding to the "correct" direction of circulation. The lower branch is unstable with respect to one perturbation, and the branch arising out of equilibrium is unstable with respect to two perturbations. In this respect the situation differs from the case of a homogeneous liquid, in which the lower branch is stable.

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